Period 1, Oct 25, 2024

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81. $F(x) = x^3$ $\left(\frac{1}{2}, \frac{1}{8}\right)$

$F^{-1}(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ $\left(\frac{1}{8}, \frac{1}{2}\right)$

$F'(x) = 3x^2 \Rightarrow F'\left(\frac{1}{2}\right) = 3\cdot\left(\frac{1}{2}\right)^2 = \frac{3}{4}$

$(F^{-1})'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}} = \frac{1}{3\sqrt[3]{\left(\frac{1}{8}\right)^2}} = \frac{1}{3\sqrt[3]{\frac{1}{64}}} = \frac{1}{3\cdot\frac{1}{4}}$

$\frac{4}{3} = \frac{1}{\frac{3}{4}}$

83. $F(x) = \sqrt{x-4} \Rightarrow x$ must be greater than 4 Range $y \geq 0$

$F^{-1}(x) = x^2 + 4$ $x \geq 0$ Range $y \geq 4$

$F(x) = (x-4)^{\frac{1}{2}}$ $(5, 1)$

$F'(x) = \frac{1}{2}(x-4)^{-\frac{1}{2}}\cdot 1 = \frac{1}{2\sqrt{x-4}} = F'(5) = \frac{1}{2\sqrt{5-4}} = \frac{1}{2}$

$(F^{-1})'(x) = 2x$ $(1, 5)$

$(F^{-1})'(1) = 2\cdot 1 = 2$

49, 65, 61, 67

$$\frac{d}{dx}(\arcsin 3x) = \frac{3}{\sqrt{1-(3x)^2}}$$

$u = 3x$
$u' = 3$

49. $\dfrac{\arcsin 3x}{x} = g(x)$

$$\frac{d}{dx}(\arcsin u) = \frac{u'}{\sqrt{1-u^2}}$$

$$g'(x) = \frac{\frac{3}{\sqrt{1-9x^2}}(x) - (\arcsin 3x)\cdot 1}{x^2}$$

$$\frac{\frac{3x}{\sqrt{1-9x^2}} - \frac{(\arcsin 3x)\sqrt{1-9x^2}}{1\cdot\sqrt{1-9x^2}}}{x^2} = \frac{\frac{3x-(\arcsin 3x)\sqrt{1-9x^2}}{\sqrt{1-9x^2}}}{\frac{x^2}{1}}$$

$$g'(x) = \frac{3x - \sqrt{1-9x^2}\arcsin 3x}{\sqrt{1-9x^2}} \cdot \frac{1}{x^2} = \frac{3x - \sqrt{1-9x^2}\arcsin 3x}{x^2\sqrt{1-9x^2}}$$

61. $y = \arctan x + \dfrac{x}{1+x^2}$

$2x^2 - 2x^2 = 0$

$$\frac{dy}{dx} = \frac{1\,(1+x^2)}{(1+x^2)(1+x^2)} + \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{1+\cancel{x^2}+1+\cancel{x^2}-\cancel{2x^2}}{(1+x^2)^2} = \frac{2}{(1+x^2)^2}$$

65

$y = \arctan \frac{x}{2} \quad \left(2, \frac{\pi}{4}\right) \qquad \frac{d}{dx}(\arctan u) = \frac{u'}{1+u^2}$

$u = \frac{x}{2}$

$u' = \frac{1}{2}$

$$\frac{dy}{dx} = \frac{\frac{1}{2}}{1+\left(\frac{x}{2}\right)^2} = \frac{\frac{1}{2}}{1+\left(\frac{2}{2}\right)^2} = \frac{\frac{1}{2}}{2} = \frac{1}{4} = m$$

$$y - \frac{\pi}{4} = \frac{1}{4}(x-2) = \frac{1}{4}x - \frac{1}{2}$$

$$y - \cancel{\frac{\pi}{4}} = \frac{1}{4}x - \frac{1}{2} + \frac{\pi}{4} \Rightarrow y = \frac{1}{4}x - \frac{1}{2} + \frac{\pi}{4}$$

$+ \cancel{\frac{\pi}{4}}$

67.

$y = \underbrace{4x} \arccos(x-1) \qquad \frac{d}{dx}(\arccos u) = \frac{-u'}{\sqrt{1-u^2}}$

$u = x-1$

$u' = 1$

$(1, 2\pi)$

$$\frac{dy}{dx} = 4\arccos(x-1) + 4x \cdot \frac{-1}{\sqrt{1-(x-1)^2}}$$

$$= 4\arccos(1-1) + 4(1) \cdot \frac{-1}{\sqrt{1-(1-1)^2}}$$

$$\frac{dy}{dx} = m = 4 \cdot \frac{\pi}{2} + \frac{-4}{\sqrt{1}} = 2\pi - 4 \qquad \text{Point } (1, 2\pi)$$

$x_1 \quad y_1$

$\arccos 0 = a$

$\cos a = 0$

$a = \frac{\pi}{2}$

$y - y_1 = m(x - x_1)$

$y - 2\pi = (2\pi - 4)(x - 1)$

$+2\pi \qquad\qquad +2\pi$

$y = \underbrace{2\pi x - \cancel{2\pi} - 4x}_{} + 4 + \cancel{2\pi}$

$y = (2\pi - 4)\cdot x + 4$

$\log_a b = c \Leftrightarrow a^c = b$

$\ln 1 = 0$

$\log_a 1 = 0$

$\log_a b + \log_a c = \log_a bc$

$\log_a b - \log_a c = \log_a \frac{b}{c}$

$\log_a b^c = c \log_a b$

$\ln e = 1$

$\ln_e x = y \Leftrightarrow e^y = x$

$\ln x + \ln y = \ln xy$

$\ln x - \ln y = \ln \frac{x}{y}$

$\ln x^y = y \ln x$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

Change of Base	$log_b a = \frac{\ln a}{\ln b}$	$log_2 3 = \frac{\ln 3}{\ln 2}$

$$\frac{d}{dx}[\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \qquad , u > 0$$

$$= \frac{u'}{u}$$

1. Find y' if $y = \ln(2x + 2)$

$u = 2x + 2$

$u' = 2$

$$\frac{dy}{dx} = \frac{1}{2x+2}(2) = \frac{\cancel{2}}{\cancel{2}(x+1)} = \frac{1}{x+1}$$

2. Let $f(x) = \ln(\tan x)$. Find $f'(x)$

$u = \tan x$

$u' = \sec^2 x$

$$F'(x) = \frac{1}{\tan x} \cdot \sec^2 x = \frac{\cancel{\cos x}}{\sin x} \cdot \frac{1}{\cos^{\cancel{2}} x} = \frac{1}{\sin x} \cdot \frac{1}{\cos x}$$

$$= \csc x \cdot \sec x$$

3. $\frac{d}{dx}[\ln(x^2+1)] = \frac{1}{x^2+1}\cdot 2x = \frac{2x}{x^2+1}$

$u = x^2+1$

$u' = 2x$

_________ Rule

4. $\frac{d}{dx}[\underbrace{x}\,\underbrace{\ln x}] = 1\cdot\ln x + x\cdot\frac{1}{x}$

$=(\ln x)+1$

Chain Rule

5. $\frac{d}{dx}\left[(\ln x)^3\right]$

$u = \ln x \Rightarrow y = u^3$

$\frac{du}{dx} = \frac{1}{x} \qquad \frac{dy}{dx} = 3u^2$

$\frac{\cancel{du}}{dx}\cdot\frac{dy}{\cancel{du}} = \frac{dy}{dx}$

$\frac{1}{x}\cdot 3u^2 = \frac{1}{x}\cdot 3(\ln x)^2 = \frac{3(\ln x)^2}{x}$

Example Set B: Expanding Logarithmic Expressions

1) $\ln\left(\dfrac{10}{9}\right) = \ln 10 - \ln 9$

2) $\ln\sqrt{3x+2} = \ln(3x+2)^{\frac{1}{2}} = \frac{1}{2}\ln(3x+2) = \dfrac{\ln(3x+2)}{2}$

3) $\ln\dfrac{6x}{5} = \ln 6x - \ln 5$

Example Set C

1. $\dfrac{d}{dx}\left[\ln\sqrt{x+1}\right] = \dfrac{d}{dx}\left[\ln(x+1)^{\frac{1}{2}}\right] = \dfrac{d}{dx}\left[\frac{1}{2}\ln(x+1)\right]$

$$= \frac{1}{2}\cdot\frac{1}{x+1}\cdot 1 = \frac{1}{2(x+1)}$$

2. $$\frac{d}{dx}\left(\ln\frac{x\left(x^2+1\right)^2}{\sqrt{2x^3-1}}\right)$$

$$\frac{d}{dx}\left[\ln x + 2\ln(x^2+1) - \frac{1}{2}\ln(2x^3-1)\right]$$

$$\frac{1}{x} + 2\cdot\frac{1}{x^2+1}\cdot 2x - \frac{1}{2}\cdot\frac{1}{2x^3-1}\cdot 6x^2$$

$$\frac{1}{x} + \frac{4x}{x^2+1} - \frac{\cancel{6}^{3}x^2}{\cancel{2}(2x^3-1)}$$

$$\log_a b = c \Rightarrow a^c = b \Rightarrow \ln a^c = \ln b$$

$$\frac{c\,\cancel{\ln a}}{\cancel{\ln a}} = \frac{\ln b}{\ln a}$$

$$\log_a b = \frac{\ln b}{\ln a}$$

$$c = \frac{\ln b}{\ln a}$$

$$\log_a x = \frac{1}{\ln a}\ln x$$

1. Re write $f(x)$ using the properties of logs and find $f'(x)$

$$f(x) = \log_5 \sqrt{x} = \frac{\ln \sqrt{x}}{\ln 5} = \left(\frac{1}{\ln 5}\right)\left(\ln x^{\frac{1}{2}}\right) = \frac{1}{2\ln 5} \cdot \ln x$$

$$F'(x) = \underbrace{\frac{1}{2\ln 5} \cdot \frac{1}{x}}_{\frac{1}{2x\ln 5}} = \frac{1}{x\ln 25} = \frac{1}{\ln 5^{2x}} = \frac{1}{\ln 25^{x}}$$

Derivatives for Log Functions of Base a
Let $\boldsymbol{a}$ be a positive real number $(\boldsymbol{a} \neq \mathbf{1})$ and let $\boldsymbol{u}$ be a differentiable function of $\boldsymbol{x}$ $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$

More Generally….

$$\frac{d}{dx}[\log_a u] = \frac{u'}{(\ln a)u}$$

Example Set E #1

$$y = \log_7 x$$ $\Rightarrow \frac{1}{x \ln 7} = \frac{1}{\ln 7^x}$

Example Set E #2

$$y = \log_7(5x^3 + 6)$$ $\Rightarrow \frac{dy}{dx} = \frac{15x^2}{(5x^3+6)\ln 7}$

Example Set E #3

$$y = \log_7(\tan(\ln x^2))$$

$y = \log_7 (\tan(2\ln x))$

$u = 2\ln x \qquad L = \tan u$

$\frac{dy}{dx} = \frac{(\sec^2(\ln x^2)) \cdot \frac{2}{x}}{(\ln 7)(\tan(\ln x^2))}$

By utilizing the rules of logarithms and implicit differentiation, you can turn an exponential equation into an equation involving logarithms that is usually easier to deal with.

Example

1. $\frac{d}{dx}[2^x]$

$$y = 2^x$$

$$\ln y = \ln 2^x$$

$$\ln y = x \underbrace{\ln 2}_{\text{constant}}$$

$$\cancel{y} \cdot \frac{1}{\cancel{y}}\frac{dy}{dx} = \ln 2 \cdot y$$

$$\frac{dy}{dx} = y \ln 2 = 2^x \cdot \ln 2$$

Derivatives of a^x
Let a be a constant $$\frac{d}{dx}[a^x] = \ln a \cdot a^x$$

More Generally….

$$\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$$

$$= (\ln a)a^u \cdot u'$$

Example Set F: Find the Derivative

1. $y = 3^x$

$\frac{dy}{dx} = (\ln 3)\cdot 3^x \cdot 1$

2. $y = 2^{3x}$

$\frac{dy}{dx} = (\ln 2)\cdot 2^{3x} \cdot 3$

$(3\ln 2)2^{3x}$

$(\ln 8)2^{3x}$

Example Set F: Find the Derivative

3. $y = 7^{\sin(2-\pi x)}$

$\frac{dy}{dx} = (\ln 7)\cdot 7^{\sin(2-\pi x)} \cdot (\cos(2-\pi x))\cdot -\pi$

4. $y = 7^{\csc(x^2)}$

$\frac{dy}{dx} = (\ln 7)\, 7^{\csc(x^2)} \cdot (-\csc x^2 \cot x^2)(2x)$

Find the derivative of $f(x) = e^{5x} + 7^{2x} + \ln(x^2 + 4)$

$F'(x) = e^{5x}\cdot 5 + (\ln 7)\cdot 7^{2x}\cdot 2 + \frac{1}{x^2+4}\cdot 2x$

$F'(x) = 5e^{5x} + (2\ln 7)7^{2x} + \frac{2x}{x^2+4}$

Find the derivative of $f(x) = e^{\tan 3x} + 6^{x^2} + \ln(\sec x)$

$F'(x) = 3\sec^2 3x \cdot e^{\tan 3x} + (2x\ln 6)6^{x^2} + \frac{1}{\cancel{\sec x}} \cdot \cancel{\sec x}\tan x$

1. Find the y' if $y = x^x$

$\ln y = \ln x^x = \underbrace{x}\,\underbrace{\ln x}$

$\cancel{y} \cdot \frac{1}{\cancel{y}} \cdot \frac{dy}{dx} = \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) \cdot y$

$\frac{dy}{dx} = (\ln x + 1)\,x^x = x^x \ln x + x^x$

$$y = \frac{x\sqrt{x^2+1}}{(x+1)^{\frac{2}{3}}}$$

$$\ln y = \ln \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}$$

$$\ln y = \ln x + \tfrac{1}{2}\ln(x^2+1) - \tfrac{2}{3}\ln(x+1)$$

$$y \cdot \frac{1}{\cancel{y}}\frac{dy}{dx} = \left(\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x - \frac{2}{3}\cdot\frac{1}{x+1}\cdot 1\right) y$$

$$\frac{dy}{dx} = \frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x} + \frac{1\cdot 2x}{2(x^2+1)} - \frac{2}{3(x+1)}\right)$$

$$y = \sqrt[3]{\frac{(2x+3)^2(x-2)^4}{(x+1)}}$$

$$\ln y = \ln\left(\frac{(2x+3)^2(x-2)^4}{(x+1)}\right)^{\frac{1}{3}} = \frac{1}{3}\left[\ln(2x+3)^2 + \ln(x-2)^4 - \ln(x+1)\right]$$

$$\ln y = \frac{1}{3}\left[2\ln(2x+3) + 4\ln(x-2) - \ln(x+1)\right.$$

$$y \cdot \frac{1}{\cancel{y}}\frac{dy}{dx} = \frac{1}{3}\left\{2\cdot\frac{1}{2x+3}\cdot 2 + 4\cdot\frac{1}{x-2}\cdot 1 - \frac{1}{x-1}\cdot 1\right]\cdot y$$

$$\frac{dy}{dx} = \frac{1}{3}\left[\frac{4}{2x+3} + \frac{4}{x-2} - \frac{1}{x}\right]\left[\sqrt[3]{\frac{(2x+3)^2(x-2)^4}{(x+1)}}\right]$$